Problem Set 6 due April 15, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider a length 1 vector $\boldsymbol{a} \in \mathbb{R}^n$ (so $||\boldsymbol{a}|| = 1$), and look at the linear transformation:

$$\phi : \mathbb{R}^n \to \mathbb{R}^n \qquad \text{corresponding to the matrix} \qquad A = I - 2\boldsymbol{a}\boldsymbol{a}^T$$

(1) Compute $\boldsymbol{a}^T\boldsymbol{a}$ and show that the matrix A is orthogonal. *(10 points)*

(2) In terms of $\boldsymbol{a}$, what is the subspace of $\mathbb{R}^n$ fixed by ϕ, i.e. the subspace: *(5 points)*

$$\{\boldsymbol{v} \in \mathbb{R}^n \text{ such that } \phi(\boldsymbol{v}) = \boldsymbol{v}\}$$

(3) Compute $\phi(\boldsymbol{a})$ and describe the linear transformation ϕ geometrically (i.e. say what it is called in plain English, and draw a picture in the $n = 3$ case). *(10 points)*

Problem 2:

Consider the function:

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y + 2 \\ 3x - y - 1 \end{bmatrix}$$

(1) Explain why f is not a linear transformation. *(5 points)*

(2) Find a linear transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and a translation $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$f = \phi \circ g$$

(a translation is a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $g(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{a}$ for a fixed vector $\boldsymbol{a}$). *(10 points)*

Problem 3:

Consider the linear transformation:

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \qquad f(\boldsymbol{v}) = A\boldsymbol{v} \qquad \text{where} \qquad A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

(1) Find a basis $\boldsymbol{w}_1, \boldsymbol{w}_2$ of $\mathbb{R}^2$ such that $f(\boldsymbol{e}_1) = \boldsymbol{w}_1$ and $f(\boldsymbol{e}_2) = \boldsymbol{w}_2$, where $\boldsymbol{e}_i$ is the i–th coordinate unit vector. Compute the matrix B which represents f in the new basis, i.e.:

$$f(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) = (b_{11}x_1 + b_{12}x_2 + b_{13}x_3)\boldsymbol{w}_1 + (b_{21}x_1 + b_{22}x_2 + b_{23}x_3)\boldsymbol{w}_2$$

and say explicitly how B relates to A (*Hint: B should be equal to a matrix times A*). *(10 points)*

(2) Construct a basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ of $\mathbb{R}^3$ such that $f(\boldsymbol{v}_1) = \boldsymbol{e}_1$, $f(\boldsymbol{v}_2) = \boldsymbol{e}_2$, $f(\boldsymbol{v}_3) = 0$, where $\boldsymbol{e}_i$ is the i–th coordinate unit vector. Compute the matrix C which represents f in the new basis, i.e.:

$$f(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3) = (c_{11}x_1 + c_{12}x_2 + c_{13}x_3)\boldsymbol{e}_1 + (c_{21}x_1 + c_{22}x_2 + c_{23}x_3)\boldsymbol{e}_2$$

and say explicitly how C relates to A (*Hint: C should be equal to A times a matrix*). *(10 points)*

Problem 4:

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

(1) Compute $\det A$ by row operations (i.e. $\pm$ product of pivots). *(10 points)*

(2) Compute $\det A$ by cofactor expansion. *(10 points)*

Note: you may use an explicit formula for 2×2 determinants, but not for bigger ones.

Problem 5:

Consider the matrix:

$$A_n = \begin{bmatrix} 0 & x_1 & x_2 & \dots & x_n \\ x_1 & 1 & 0 & \dots & 0 \\ x_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \dots & 1 \end{bmatrix}$$

Compute a recursive formula for $\det A_n$, and then obtain an explicit formula. *(20 points)*